

ON SYMPLECTIC HALF-FLAT MANIFOLDS

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ABSTRACT. We construct examples of symplectic half-flat manifolds on compact quotients of solvable Lie groups. We prove that the Calabi-Yau structures are not rigid in the class of symplectic half-flat structures. Moreover, we provide an example of a compact 6-dimensional symplectic half-flat manifold whose real part of the complex volume form is d -exact. Finally we discuss the 4-dimensional case.

1. INTRODUCTION

Half-flat structures arise as a special class of $SU(3)$ -structures introduced by Hitchin in [13]. An $SU(3)$ -structure (ω, J, ψ) on a 6-dimensional manifold M is said to be *half-flat* if the defining forms $\omega \in \Lambda^2(M)$, $\Re \psi \in \Lambda^3(M)$ satisfy

$$(1) \quad d\omega \wedge \omega = 0, \quad d\Re \psi = 0.$$

Condition (1) is equivalent to require that the intrinsic torsion of (ω, J, ψ) is symmetric. In [13] Hitchin proves that, starting with a half-flat manifold (M, ω, J, ψ) , if certain evolution equations have a solution coinciding with the initial datum (ω, ψ) at time $t = 0$, then there exists a metric with holonomy contained in G_2 on $M \times I$ for some interval I (see also [2], [1]).

In the present paper we study *symplectic half-flat manifolds*, namely 6-dimensional manifolds M endowed with an $SU(3)$ -structure (ω, J, ψ) satisfying the following¹

$$(2) \quad d\omega = 0, \quad d\Re \psi = 0, \quad \psi \wedge \bar{\psi} = -\frac{4}{3}i\omega^3.$$

In such a case, since $\Re \psi$ is a calibration on M , we can define *special Lagrangian submanifolds* as compact three-dimensional Lagrangian submanifolds of M , calibrated by $\Re \psi$ (see [12]).

It turns out (see lemma 2.3 and [6] for its proof) that the complex volume form ψ of a symplectic half-flat manifold (M, ω, J, ψ) is parallel with respect to the Chern connection ∇ of the almost Kähler manifold (M, ω, J) . Therefore, in such a case, the holonomy of ∇ is contained in $SU(3)$. Note that if the almost complex structure J is integrable, then (M, ω, J) is a Kähler manifold and the Chern connection coincides with the Levi-Civita one and, consequently, in this case symplectic half-flat

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¹In [6, 7] symplectic half-flat manifolds were called *special generalized Calabi-Yau manifolds*. Here we change the terminology to avoid confusion with the case considered by Hitchin in [14].

manifolds are Calabi-Yau manifolds of complex dimension 3. Therefore, symplectic half-flat manifolds can be viewed as an extension of Calabi-Yau manifolds to the non-integrable case. In particular, symplectic half-flat manifolds are *generalized Calabi-Yau manifolds* in the sense of Hitchin. Indeed, Hitchin in [14] gives the notion of *generalized Calabi-Yau structure* on a manifold M of dimension $2m$ as a closed complex form φ of mixed degree which is a complex pure spinor for the orthogonal vector bundle $TM \oplus T^*M$ endowed with the natural pairing \langle, \rangle and such that $\langle \varphi, \overline{\varphi} \rangle \neq 0$. It turns out that such a structure (see [14]) induces a *generalized complex structure* on M (see [11]). Two basic examples of generalized Calabi-Yau manifolds are furnished by holomorphic Calabi-Yau manifolds and symplectic manifolds. In the first case if Ψ denotes the holomorphic volume form on the Kähler manifold M , then $\varphi = \Psi$ defines a generalized Calabi-Yau structure on M . In the second case, if ω is the symplectic form on M , then $\varphi = \exp(i\omega)$ is a generalized Calabi-Yau structure on M .

In this paper we give some explicit examples of compact symplectic half-flat manifolds arising as quotient of solvable Lie groups and we construct a family of symplectic half-flat structures (ω_t, J_t, ψ_t) on the 6-dimensional torus \mathbb{T}^6 , coinciding with the standard Calabi-Yau structure for $t = 0$, but which is not Calabi-Yau for $t \neq 0$ (see example 3.2 and theorem 3.3). For other examples we refer to [7] and [3]. Then we give a symplectic half-flat structure on a compact 6-manifold whose complex volume form has real part d -exact (see theorem 3.5). This is in contrast with the integrable case, namely in the context of Calabi-Yau geometry, where the real part of the complex volume form cannot be exact. Finally, we consider the 4-dimensional case. As in the 6-dimensional case, we characterize such structures in terms of stable forms (see proposition 4.2).

In section 2, we start by recalling some facts on symplectic half-flat manifolds. In section 3 we describe the compact examples previous mentioned. In the last section we consider the 4-dimensional case.

2. SYMPLECTIC HALF-FLAT GEOMETRY

Let M be a 6-dimensional smooth manifold and let $L(M)$ be the principal $GL(6, \mathbb{R})$ -bundle of linear frames on M . An $SU(3)$ -structure on M is a reduction of $L(M)$ to a principal bundle whose structure group is isomorphic to $SU(3)$. It is well known that $SU(3)$ -structures on M are in one-to-one correspondence with the triple (ω, J, ψ) , where

- ω is a non-degenerate 2-form on M ;
- J is an ω -calibrated almost complex structure, i.e. J is an almost complex structure on M such that the tensor $g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)$ is an almost Hermitian metric.
- $\psi \in \Lambda_J^{3,0}(M)$ is a complex volume form on M such that

$$\psi \wedge \overline{\psi} = -\frac{4}{3}i\omega^3.$$

Definition 2.1. An $SU(3)$ -structure (ω, J, ψ) , is said to be symplectic half-flat if

$$d\omega = 0, \quad d\Re \psi = 0.$$

Symplectic half-flat manifolds lie in the intersection of symplectic manifolds and half-flat manifolds. The latter ones have been introduced by Hitchin in [13] (see also [2]).

Now we present symplectic half-flat structures in terms of differential forms and the Chern connection. In order to do this we start with recalling the following

Definition 2.2. *Let (V, ω) be a symplectic vector space of dimension $2n$; the symplectic Hodge operator*

$$\star: \Lambda^r(V^*) \rightarrow \Lambda^{2n-r}(V^*)$$

is defined by

$$\alpha \wedge \star \beta = \omega(\alpha, \beta) \frac{\omega^n}{n!}.$$

Now we describe the standard model:

On \mathbb{R}^6 let us consider the natural symplectic structure

$$\omega_3 = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6,$$

where $\{x_1, \dots, x_6\}$ are the standard coordinates on \mathbb{R}^6 , and let

$$\Lambda_0^3(\mathbb{R}^{6*}) = \{\Omega \in \Lambda^3(\mathbb{R}^{6*}) \mid \Omega \wedge \omega_3 = 0\}.$$

The group

$$G = \mathrm{Sp}(3, \mathbb{R}) \times \mathbb{R}_+^*$$

acts on $\Lambda_0^3(\mathbb{R}^{6*})$ by

$$(A, t)(\Omega) = tA^*(\Omega),$$

for any $A \in \mathrm{Sp}(3, \mathbb{R})$, $t \in \mathbb{R}_+^*$, where $\mathrm{Sp}(3, \mathbb{R})$ denotes the symplectic group on (\mathbb{R}^6, ω_3) .

Moreover, the 3-form

$$\Omega_0 = \Re(dz_1 \wedge dz_2 \wedge dz_3)$$

belongs to $\Lambda_0^3(\mathbb{R}^{6*})$, where $dz_h = dx_h + i dx_{h+3}$, $h = 1, 2, 3$.

Let us consider the map

$$\Lambda_0^3(\mathbb{R}^{6*}) \rightarrow \mathrm{End}(\Lambda^1(\mathbb{R}^{6*}))$$

defined by

$$P_\Omega(\alpha) = -\frac{1}{2} \star(\Omega \wedge \star(\Omega \wedge \alpha)),$$

for any $\alpha \in \Lambda^1(\mathbb{R}^{6*})$. We have

- $\omega_3(P_\Omega \alpha, \beta) = -\omega_3(\alpha, P_\Omega \beta)$,
- $P_\Omega^2 = cI$, $c \in \mathbb{R}$

(see e.g. [6]).

Given $\Omega \in \Lambda_0^3(\mathbb{R}^{6*})$, let

$$F_\Omega: \Lambda^1(\mathbb{R}^{6*}) \rightarrow \Lambda^4(\mathbb{R}^{6*})$$

$$F_\Omega(\alpha) = \Omega \wedge \alpha.$$

Then we get that the following facts are equivalent:

- Ω belongs to the G -orbit of Ω_0 ,
- F_Ω is injective and ω_3 is negative defined on $\mathrm{Im}(F_\Omega)$.

A 3-form $\Omega \in \Lambda^3(\mathbb{R}^{6*})$ is said to be *positive* if F_Ω is injective and ω_3 is negative defined on $\text{Im}(F_\Omega)$ (in this case $c = -\sqrt[3]{\det P_\Omega}$) and *normalized* if $\det P_\Omega = 1$.

Let (M, ω) be a symplectic manifold. Let J be an ω -calibrated almost complex structure on M and let

$$g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)$$

be the almost Hermitian metric associated with (ω, J) . Denote by ∇^{LC} the Levi-Civita connection of g_J . Then the *Chern connection* on (M, ω, J) is defined by

$$\nabla = \nabla^{LC} - \frac{1}{2}J\nabla^{LC}J.$$

It is known that

$$\nabla g = 0, \quad \nabla J = 0, \quad T^\nabla = \frac{1}{4}N_J$$

where

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

is the *Nijenhuis tensor* of J (see e.g. [10]). Let (M, J) be an almost complex manifold; then the exterior derivative

$$d: \Lambda_J^{p,q}(M) \rightarrow \Lambda_J^{p+2,q-1}(M) \oplus \Lambda_J^{p+1,q}(M) \oplus \Lambda_J^{p,q+1}(M) \oplus \Lambda_J^{p-1,q+2}(M)$$

splits as

$$d = A_J + \partial_J + \bar{\partial}_J + \bar{A}_J.$$

Note that by the Newlander-Nirenberg theorem J is complex if and only if $A_J = 0$.

Let (M, ω) be a 6-dimensional (compact) manifold equipped with an ω -calibrated almost complex structure J and a complex $(3, 0)$ -form ψ satisfying

$$\psi \wedge \bar{\psi} = -ie^\sigma \omega^3,$$

where σ is a C^∞ function on M . Set

$$\Omega := \Re \psi.$$

The following lemma (see [6]) gives a characterization of symplectic half-flat structures in terms of differential forms and the Chern connection.

Lemma 2.3. *With the notation above, the following facts are equivalent*

- a) $\begin{cases} d\Omega = 0 \\ \sigma = \text{const.} \end{cases}$
- b) $\begin{cases} d\Omega = 0 \\ \Omega \wedge \omega = 0 \\ \frac{2}{3}\sqrt{3}e^{-\sigma/2}\Omega \text{ is positive and normalized at any point} \end{cases}$
- c) $\begin{cases} \nabla\psi = 0 \\ A_J(\bar{\psi}) + \bar{A}_J(\psi) = 0. \end{cases}$

Remark 2.4. By lemma 2.3 one can define a symplectic half-flat manifold as a (compact) 6-dimensional symplectic manifold (M, ω) endowed with an ω -calibrated almost complex structure J and a $(3, 0)$ -form ψ such that

$$\begin{cases} \nabla\psi = 0 \\ A_J(\bar{\psi}) + \bar{A}_J(\psi) = 0. \end{cases}$$

Therefore, the holonomy of the Chern connection of a symplectic half-flat manifold is contained in $SU(3)$.

Alternatively, a symplectic half-flat structure can be given by a pair (ω, Ω) , where ω is a symplectic form and Ω is a closed 3-form satisfying $\Omega \wedge \omega = 0$ and which is positive and normalized at any point. Indeed, in this case the ω -calibrated almost complex structure is given by the isomorphism dual to P_Ω and the complex volume form is

$$\psi := \Omega + iP_\Omega(\Omega).$$

Remark 2.5. Observe that if

$$d\Re \psi = d\Im \psi = 0,$$

then the almost complex structure is integrable (hence (M, ω, J, ψ) is a Calabi-Yau manifold).

Indeed, if $\alpha \in \Lambda_J^{1,0}(M)$, then

$$0 = d(\alpha \wedge \psi) = d\alpha \wedge \psi$$

and consequently

$$d(\Lambda_J^{1,0}(M)) \subset \Lambda_J^{2,0}(M) \oplus \Lambda_J^{1,1}(M).$$

Let (M, ω, J, ψ) be a symplectic half-flat manifold. As in the Calabi-Yau case, the 3-form $\Omega = \Re \psi$ is a calibration on M (see [12]). Therefore we can give the following

Definition 2.6. A special Lagrangian submanifold of (M, ω, J, ψ) is a compact submanifold $p: L \hookrightarrow M$ calibrated by $\Re \psi$.

We have the following

Lemma 2.7. Let $p: L \hookrightarrow M$ be a submanifold. The following facts are equivalent

1. $p^*(\omega) = 0$, $p^*(\Im \psi) = 0$;
2. there exists an orientation on L making it calibrated by $\Re \psi$.

3. EXAMPLES OF COMPACT SYMPLECTIC HALF-FLAT SOLVMANIFOLDS

In this section we give some examples symplectic half-flat manifolds and special Lagrangian submanifolds. We give also an example of a smooth family of a symplectic half-flat structures on the 6-dimensional torus which is integrable for $t = 0$, but not integrable for $t \neq 0$.

Example 3.1. Let G be the Lie group of matrices of the form

$$A = \begin{pmatrix} e^t & 0 & xe^t & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & xe^{-t} & 0 & y_2 \\ 0 & 0 & e^t & 0 & 0 & w_1 \\ 0 & 0 & 0 & e^{-t} & 0 & w_2 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let

$$(3) \quad \begin{aligned} \alpha_1 &= dt, \quad \alpha_2 = dx, \quad \alpha_3 = e^{-t}dy_1 - xe^{-t}dw_1 \\ \alpha_4 &= e^tdy_2 - xe^tdw_2, \quad \alpha_5 = e^{-t}dw_1, \quad \alpha_6 = e^tdw_2. \end{aligned}$$

Then $\{\alpha_1, \dots, \alpha_6\}$ is a basis of left-invariant 1-forms.

By (3) we easily get

$$(4) \quad \begin{cases} d\alpha_1 = d\alpha_2 = 0 \\ d\alpha_3 = -\alpha_1 \wedge \alpha_3 - \alpha_2 \wedge \alpha_5 \\ d\alpha_4 = \alpha_1 \wedge \alpha_4 - \alpha_2 \wedge \alpha_6 \\ d\alpha_5 = -\alpha_1 \wedge \alpha_5 \\ d\alpha_6 = \alpha_1 \wedge \alpha_6. \end{cases}$$

Let $\{\xi_1, \dots, \xi_6\}$ be the dual frame of $\{\alpha_1, \dots, \alpha_6\}$; we have

$$(5) \quad \begin{aligned} \xi_1 &= \frac{\partial}{\partial t}, \quad \xi_2 = \frac{\partial}{\partial x}, \quad \xi_3 = e^t \frac{\partial}{\partial y_1}, \quad \xi_4 = e^{-t} \frac{\partial}{\partial y_2} \\ \xi_5 &= e^t \frac{\partial}{\partial w_1} + x e^t \frac{\partial}{\partial y_1}, \quad \xi_6 = e^{-t} \frac{\partial}{\partial w_2} + x e^{-t} \frac{\partial}{\partial y_2}. \end{aligned}$$

From (5) we obtain

$$(6) \quad \begin{aligned} [\xi_1, \xi_3] &= \xi_3, \quad [\xi_1, \xi_4] = -\xi_4, \quad [\xi_1, \xi_5] = \xi_5 \\ [\xi_1, \xi_6] &= -\xi_6, \quad [\xi_2, \xi_5] = \xi_3, \quad [\xi_2, \xi_6] = \xi_4 \end{aligned}$$

and the other brackets are zero.

Therefore G is a non-nilpotent solvable Lie group.

By [8] G has a cocompact lattice Γ . Hence

$$M = G/\Gamma$$

is a compact solvmanifold of dimension 6. Let us denote with $\pi: \mathbb{R}^6 \rightarrow M$ the natural projection. Define

$$\omega = \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5$$

and

$$\begin{aligned} J(\xi_1) &= \xi_2, \quad J(\xi_3) = \xi_6, \quad J(\xi_4) = \xi_5 \\ J(\xi_2) &= -\xi_1, \quad J(\xi_6) = -\xi_3, \quad J(\xi_5) = -\xi_6. \end{aligned}$$

Then ω is a symplectic form on M and J is an ω -calibrated almost complex structure on M . Set

$$\psi = i(\alpha_1 + i\alpha_2) \wedge (\alpha_3 + i\alpha_6) \wedge (\alpha_4 + i\alpha_5);$$

a direct computation shows that (ω, J, ψ) is a symplectic half-flat structure on M .

Let consider now the lattice $\Sigma \subset \mathbb{R}^4$ given by

$$\Lambda := \text{Span}_{\mathbb{Z}} \left\{ \begin{pmatrix} -\mu \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \mu \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\mu \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \mu \end{pmatrix} \right\},$$

where $\mu = \frac{\sqrt{5}-1}{2}$. Let \mathbb{T}^4 be the torus

$$\mathbb{T}^4 = \mathbb{R}^4 / \Lambda.$$

For any $p, q \in \mathbb{Z}$ let $\rho(p, q)$ be the transformation of \mathbb{T}^4 represented by the matrix

$$\begin{pmatrix} e^{p\lambda} & 0 & qe^{p\lambda} & 0 \\ 0 & e^{-p\lambda} & 0 & qe^{-p\lambda} \\ 0 & 0 & e^{p\lambda} & 0 \\ 0 & 0 & 0 & e^{-p\lambda} \end{pmatrix},$$

where $\lambda = \log \frac{3+\sqrt{5}}{2}$.

Then

$$A(p, q)([y_1, y_2, z_1, z_2], (t, x)) = (\rho(p, q)[y_1, y_2, z_1, z_2], (t + p, x + q))$$

is a transformation of $\mathbb{T}^4 \times \mathbb{R}^2$ for any $p, q \in \mathbb{Z}$. Let Θ be the group of such transformations. The manifold M can be identified with

$$(7) \quad \frac{\mathbb{T}^4 \times \mathbb{R}^2}{\Theta}$$

(see [8]).

Let consider now the involutive distribution \mathcal{D} generated by $\{\xi_2, \xi_3, \xi_4\}$ and let $p: L \hookrightarrow M$ be the leaf through $\pi(0)$.

By (5) and the identification (7) we get

$$\pi^{-1}(L) = \{x = (x_1, \dots, x_6) \in \mathbb{R}^6 \mid x_1 = x_5 = x_6 = 0\};$$

hence L is a compact submanifold of M . By a direct computation one can check that

$$\begin{cases} p^*(\omega) = 0, \\ p^*(\Im \psi) = 0. \end{cases}$$

Hence L is a special Lagrangian submanifold of (M, ω, J, ψ) . Moreover, by [8] we have

$$H^2(M, \mathbb{R}) = \text{Span}_{\mathbb{R}}\{[\alpha_1 \wedge \alpha_2], [\alpha_5 \wedge \alpha_6], [\alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5]\}$$

Therefore, we get

$$p^*(H^2(M, \mathbb{R})) = 0.$$

Example 3.2. Let (x_1, \dots, x_6) be coordinates on \mathbb{R}^6 and let

$$\omega_3 = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6$$

be the standard symplectic form on \mathbb{R}^6 . Let $a = a(x_1)$, $b = b(x_2)$, $c = c(x_3)$ be three smooth functions such that

$$\lambda_1 := b(x_2) - c(x_3), \quad \lambda_2 := -a(x_1) + c(x_3), \quad \lambda_3 = a(x_1) - b(x_2)$$

are \mathbb{Z}^6 -periodic. Let us consider the ω_3 -calibrated complex structure on \mathbb{R}^6 defined by

$$\begin{cases} J(\frac{\partial}{\partial x_r}) &= e^{-\lambda_r} \frac{\partial}{\partial x_{3+r}} \\ J(\frac{\partial}{\partial x_{3+r}}) &= -e^{\lambda_r} \frac{\partial}{\partial x_r} \end{cases}$$

$r = 1, 2, 3$. Define a $(3,0)$ -form on \mathbb{R}^6 by

$$\psi = i(dx_1 + ie^{\lambda_1} dx_4) \wedge (dx_2 + ie^{\lambda_2} dx_5) \wedge (dx_3 + ie^{\lambda_3} dx_6).$$

Then we get

$$\begin{cases} \psi \wedge \bar{\psi} = -i \frac{4}{3} \omega_3^3 \\ d\Re \psi = 0. \end{cases}$$

Since $\lambda_1, \lambda_2, \lambda_3$ are \mathbb{Z}^6 -periodic, (ω_3, J, ψ) defines a symplectic half-flat structure on the torus $\mathbb{T}^6 = \mathbb{R}^6 / \mathbb{Z}^6$. Now consider the three-torus $L = \pi(X)$, where $\pi: \mathbb{R}^6 \rightarrow \mathbb{T}^6$ is the natural projection and

$$X = \{(x_1, \dots, x_6) \in \mathbb{R}^6 \mid x_1 = x_2 = x_3 = 0\}.$$

It is immediate to check that L is a special Lagrangian submanifold of \mathbb{T}^6 .

Now we are ready to state the following

Theorem 3.3. *There exists a family (ω_t, J_t, ψ_t) of symplectic half-flat structures on the 6-dimensional torus \mathbb{T}^6 , such that (ω_0, J_0, ψ_0) is the standard Calabi-Yau structure, but (ω_t, J_t, ψ_t) is not integrable for $t \neq 0$.*

Proof. By using the notation of example 3.2, let

$$\begin{cases} J(\frac{\partial}{\partial x_r}) &= e^{-t\lambda_r} \frac{\partial}{\partial x_{3+r}} \\ J(\frac{\partial}{\partial x_{3+r}}) &= -e^{t\lambda_r} \frac{\partial}{\partial x_r}, \end{cases}$$

for $r = 1, 2, 3$,

$$\omega_t = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6$$

and

$$\psi_t = i(dx_1 + ie^{t\lambda_1} dx_4) \wedge (dx_3 + ie^{t\lambda_2} dx_5) \wedge (dx_3 + ie^{t\lambda_3} dx_6).$$

Then $(\mathbb{T}^6, \omega_t, J_t, \psi_t)$ is a symplectic half-flat manifold for any $t \in \mathbb{R}$, such that $(\mathbb{T}^6, \omega_0, J_0, \psi_0)$ is the standard holomorphic Calabi-Yau torus and J_t is non-integrable for $t \neq 0$ (here we assume that $\lambda_1, \lambda_2, \lambda_3$ are not constant). \square

Example 3.4. Let consider now the Lie group G of matrices of the form

$$A = \begin{pmatrix} 1 & 0 & x_1 & u_1 & 0 & 0 \\ 0 & 1 & x_2 & u_2 & 0 & 0 \\ 0 & 0 & 1 & y & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where x_1, x_2, u_1, u_2, y, t are real numbers. Let Γ be the subgroup G formed by the matrices having integral entries. Since Γ is a cocompact lattice of G , then $M := G/\Gamma$ is a 6-dimensional nilmanifold.

Let consider

$$\begin{aligned} \xi_1 &= \frac{\partial}{\partial y} + x_1 \frac{\partial}{\partial u_1} + x_2 \frac{\partial}{\partial u_2}, & \xi_2 &= \frac{\partial}{\partial x_2}, \\ \xi_3 &= \frac{\partial}{\partial x_1}, & \xi_4 &= \frac{\partial}{\partial t}, & \xi_5 &= \frac{\partial}{\partial u_1}, & \xi_6 &= \frac{\partial}{\partial u_2}. \end{aligned}$$

Then $\{\xi_1, \dots, \xi_6\}$ is a G -invariant global frame on M .

The respective coframe $\{\alpha_1, \dots, \alpha_6\}$ satisfies

$$(8) \quad \begin{cases} d\alpha_1 = d\alpha_2 = d\alpha_3 = d\alpha_4 = 0 \\ d\alpha_5 = \alpha_1 \wedge \alpha_3 \\ d\alpha_6 = \alpha_1 \wedge \alpha_2. \end{cases}$$

The symplectic half-flat structure on M is given by the symplectic form

$$\omega = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6,$$

by the ω -calibrated almost complex structure

$$\begin{aligned} J(\xi_1) &= \xi_4, & J(\xi_2) &= \xi_5, & J(\xi_3) &= \xi_6, \\ J(\xi_4) &= -\xi_1, & J(\xi_5) &= -\xi_2, & J(\xi_6) &= -\xi_3 \end{aligned}$$

and by the complex volume form

$$\psi = (\alpha_1 + i\alpha_4) \wedge (\alpha_2 + i\alpha_5) \wedge (\alpha_3 + i\alpha_6).$$

By a direct computation we get

$$\begin{aligned}\Re \psi &= \alpha_{123} - \alpha_{345} + \alpha_{246} - \alpha_{156}, \\ \Im \psi &= \alpha_{234} - \alpha_{135} + \alpha_{126} - \alpha_{456};\end{aligned}$$

Let

$$X = \{A \in G \mid y = x_2 = u_2 = 0\}$$

and

$$L = \pi(X),$$

$\pi: G \rightarrow M$ being the canonical projection. Then L is a special Lagrangian torus embedded in (M, ω, J, ψ) .

In order to obtain some cohomological obstructions to the existence of a symplectic half-flat structure (ω, J, ψ) on a compact 6-manifold M , one can ask if the cohomology class $[\Re \psi]$ is always non-trivial. Observe that in the Calabi-Yau case one has $[\Re \psi] \neq 0$. In our context we have the following

Theorem 3.5. *There exists a compact 6-dimensional manifold admitting a symplectic half-flat structure (ω, J, ψ) such that*

$$[\Re \psi] = 0.$$

Proof. Let G be the Lie group consisting of matrices of the form

$$A = \begin{pmatrix} e^{\lambda z} & 0 & 0 & x \\ 0 & e^{-\lambda z} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where x, y, z are real numbers and

$$\lambda = \log \frac{3 + \sqrt{5}}{2}.$$

Then G is a connected solvable Lie group admitting a cocompact lattice Γ (see [9]). Therefore $N = G/\Gamma$ is a 3-dimensional parallelizable solvmanifold. It can be easily showed (see [9] again) that there exists a coframe $\{\alpha_1, \alpha_2, \alpha_3\}$ on M satisfying the following structure equations

$$\begin{cases} d\alpha_1 = -\lambda\alpha_1 \wedge \alpha_3 \\ d\alpha_2 = \lambda\alpha_2 \wedge \alpha_3 \\ d\alpha_3 = 0. \end{cases}$$

Let $M = N \times N$. Then M is a compact 6-manifold admitting a coframe $\{\alpha_1, \dots, \alpha_6\}$ satisfying

$$\begin{cases} d\alpha_1 = -\lambda\alpha_1 \wedge \alpha_3 \\ d\alpha_2 = \lambda\alpha_2 \wedge \alpha_3 \\ d\alpha_3 = 0 \\ d\alpha_4 = -\lambda\alpha_4 \wedge \alpha_6 \\ d\alpha_5 = \lambda\alpha_5 \wedge \alpha_6 \\ d\alpha_6 = 0. \end{cases}$$

Let us now construct a symplectic half-flat structure on M satisfying $[\Re \psi] = 0$. Let (ω, J) be the almost Kähler structure on M given by the symplectic form

$$\omega = \alpha_1 \wedge \alpha_2 + \alpha_4 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6$$

and by the ω -calibrated almost complex structure J defined as

$$\begin{aligned} J(\xi_1) &= \xi_2, & J(\xi_3) &= \xi_6, & J(\xi_4) &= \xi_5 \\ J(\xi_2) &= -\xi_1, & J(\xi_6) &= -\xi_3, & J(\xi_5) &= -\xi_6, \end{aligned}$$

where $\{\xi_1, \dots, \xi_6\}$ is the frame on M dual to $\{\alpha_1, \dots, \alpha_6\}$.

Then the complex 3-form

$$\psi = \frac{\sqrt{2}}{2}(1-i)(\alpha_1 + i\alpha_2) \wedge (\alpha_4 + i\alpha_5) \wedge (\alpha_3 + i\alpha_6).$$

defines together with (ω, J) a symplectic half-flat structure on M .

Moreover

$$\Re \psi = \frac{\sqrt{2}}{2}(-\alpha_{134} + \alpha_{146} - \alpha_{135} - \alpha_{156} - \alpha_{234} - \alpha_{246} + \alpha_{235} - \alpha_{256})$$

and a direct computation gives

$$\Re \psi = \frac{\sqrt{2}}{2\lambda} d(\alpha_1 \wedge \alpha_4 + \alpha_1 \wedge \alpha_5 - \alpha_2 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5).$$

This ends the proof. \square

Remark 3.6. The symplectic manifold (M, ω) , described in the last example, satisfies the *hard Lefschetz condition*, i.e.

$$\begin{array}{ccc} \omega^k : \Lambda^{3-k}(M) & \rightarrow & \Lambda^{3+k}(M) \\ \alpha & \mapsto & \omega^k \wedge \alpha \end{array}$$

$k = 1, 2$ induces an isomorphism in cohomology. Indeed, it is immediate to check that

$$H^1(M, \mathbb{R}) = \text{Span}_{\mathbb{R}}\{[\alpha_3], [\alpha_6]\},$$

$$H^2(M, \mathbb{R}) = \text{Span}_{\mathbb{R}}\{[\alpha_1 \wedge \alpha_2], [\alpha_4 \wedge \alpha_5], [\alpha_3 \wedge \alpha_6]\},$$

$$H^4(M, \mathbb{R}) = \text{Span}_{\mathbb{R}}\{[\alpha_3 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6], [\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_6], [\alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \alpha_5]\},$$

$$H^5(M, \mathbb{R}) = \text{Span}_{\mathbb{R}}\{[\alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6], [\alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6]\}.$$

Therefore, if $\omega = \alpha_1 \wedge \alpha_2 + \alpha_4 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6$, then it is easy to verify that (M, ω) satisfies the hard Lefschetz condition. Hence, in view of a Proposition of Hitchin (see [14, prop. 7]) we have that (M, ω) satisfies a dd^J -lemma (see [14, def. 5] for the precise definition). Consequently, the manifold M has a symplectic half-flat structure (ω, J, ψ) such that the symplectic structure ω gives rise to a generalized Calabi-Yau structure on M satisfying the dd^J -lemma.

4. THE FOUR-DIMENSIONAL CASE

Let (M, ω) be a (compact) 4-dimensional symplectic manifold and J be an ω -calibrated almost complex structure on M . Let ψ be a nowhere vanishing $(2, 0)$ -form on M satisfying

$$\psi \wedge \bar{\psi} = 2\omega^2.$$

Then the conditions

$$\begin{cases} \nabla \psi = 0 \\ d\Re \psi = 0 \end{cases}$$

imply

$$d(\Im \psi) = 0.$$

Indeed, if $\nabla\psi = 0$, then $\bar{\partial}_J\psi = 0$ and

$$d\Re\psi = 0 \implies \bar{\partial}_J\psi + \partial_J\bar{\psi} + \bar{A}_J\psi + A_J\bar{\psi} = \bar{A}_J\psi + A_J\bar{\psi} = 0.$$

Since $\bar{A}_J\psi \in \Lambda_J^{1,2}(M)$ and $A_J\bar{\psi} \in \Lambda_J^{2,1}(M)$, we get $d\psi = 0$ which implies that J is holomorphic. In dimension 4 we adopt the following definition (see also [5])

Definition 4.1. *Let M be a (compact) 4-dimensional manifold. A special symplectic structure on M is a triple (ω, J, ψ) , where*

- ω is a symplectic form,
- J is an ω -calibrated almost complex structure on M ,
- ψ is a non-vanishing $(2, 0)$ -form satisfying

$$\begin{cases} \psi \wedge \bar{\psi} = 2\omega^2 \\ d\Re\psi = 0. \end{cases}$$

The following proposition gives a characterizations of a 4-dimensional special symplectic structure in terms of differential forms. The proof of this proposition can be obtained using the same argument as in [4, proposition 1].

Proposition 4.2. *Symplectic half-flat structures on a 4-dimensional manifold are in one-to-one correspondence to the triple $(\omega, \Omega_+, \Omega_-)$ of 2-forms on M satisfying the following properties*

1. $\omega \wedge \Omega_+ = \omega \wedge \Omega_- = \Omega_+ \wedge \Omega_- = 0$;
2. $\Omega_+ \wedge \Omega_+ = \Omega_- \wedge \Omega_- = \omega \wedge \omega \neq 0$;
3. if $\iota_X\Omega_+ = \iota_Y\Omega_-$, then $\omega(X, Y) \geq 0$;
4. $d\omega = d\Omega_+ = 0$.

Note that, when a triple $(\omega, \Omega_+, \Omega_-)$ is given, the complex volume form of the associated special symplectic structure is simply obtained by taking

$$\psi = \Omega_+ + i\Omega_-.$$

The following proposition gives an explicit formula for the almost complex structure induced by a triple of 2-forms $(\omega, \Omega_+, \Omega_-)$ satisfying the properties stated above

Proposition 4.3. *Let M be a 4-manifold and let $(\omega, \Omega_+, \Omega_-)$ be a triple of forms satisfying the properties 1 – 4 of proposition 4.2. Then the ω -calibrated almost complex structure induced by $(\omega, \Omega_+, \Omega_-)$ is the dual of the endomorphism P of T^*M defined by the following formula*

$$P(\alpha) = \star(\omega \wedge \star(\Omega_+ \wedge \alpha)),$$

where, as usual, \star is the symplectic star operator associated with ω and $\alpha \in T^*M$.

Proof. Since the triple $(\omega, \Omega_+, \Omega_-)$ induces an $SU(2)$ -structure, M admits a local coframe $\mathcal{B} = \{e^1, e^2, e^3, e^4\}$ with respect to which the structure forms $\omega, \Omega_+, \Omega_-$ take the standard expressions

$$\begin{aligned} \omega &= e^1 \wedge e^2 + e^3 \wedge e^4, \\ \Omega_+ &= e^1 \wedge e^3 - e^2 \wedge e^4, \\ \Omega_- &= e^1 \wedge e^4 + e^2 \wedge e^3. \end{aligned}$$

With respect to the dual frame of \mathcal{B} the almost complex structure J induced by $(\omega, \Omega_+, \Omega_-)$ on M reads as the standard one given by the following relations

$$\begin{aligned} J(e_1) &= e_2, & J(e_2) &= -e_1 \\ J(e_3) &= e_4, & J(e_4) &= -e_3. \end{aligned}$$

Hence it is sufficient to prove the proposition for these data and the latter is a direct computation. \square

Note that Ω_+, Ω_- and J are related by $\Omega_- = J\Omega_+$. Furthermore if $*$ denotes the Hodge star operator of the metric induced by a triple $(\omega, \Omega_+, \Omega_-)$, then $*\Omega_+ = \Omega_-$. The following lemma gives a topological obstruction to the existence of special symplectic structures on compact 4-manifolds.

Lemma 4.4. *Let M be a 4-dimensional compact manifold admitting a special symplectic structure. Then*

$$\dim(H^2(M, \mathbb{R})) \geq 2.$$

Proof. Let (ω, J, ψ) be a special symplectic structure on M and let $\Omega_+ = \Re \psi$, $\Omega_- = \Im \psi$. First of all we observe that Ω_+ is a symplectic form on M and consequently it cannot be exact. Furthermore if $a[\omega] + b[\Omega_+] = 0$ for some $a, b \in \mathbb{R}$, then

$$a\Omega_+ + b\omega = d\alpha,$$

for some $\alpha \in \Lambda^1(M)$. This last equation together with $\Omega_+ \wedge \omega = 0$ readily implies $b\omega^2 = d(\alpha \wedge \omega)$, which forces b to vanish. Hence ω and Ω_+ induce \mathbb{R} -linear independent classes in $H^2(M, \mathbb{R})$ and $\dim(H^2(M, \mathbb{R})) \geq 2$. \square

As in the 6-dimensional case, if (M, ω, J, ψ) is a 4-dimensional symplectic half-flat manifold, then $\Omega = \Re \psi$ is a calibration on M . Furthermore in an obvious way we have the definition of special Lagrangian submanifold.

We end this section with the following

Example 4.5. Now we recall the construction of the Kodaira-Thurston manifold. Let G be the Lie subgroup of $\mathrm{GL}(5, \mathbb{R})$ whose matrices have the following form

$$A = \begin{pmatrix} 1 & x & z & 0 & 0 \\ 0 & 1 & y & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z, t \in \mathbb{R}$. Let Γ be the subgroup of G of matrices with integers entries. Since Γ is a cocompact lattice in G , we get that

$$M = G/\Gamma$$

is a compact manifold. M is called the *Kodaira-Thurston manifold*.

Let $\{\xi_1, \dots, \xi_4\}$ be the global frame of M given by

$$\xi_1 = \frac{\partial}{\partial x}, \quad \xi_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad \xi_3 = \frac{\partial}{\partial z}, \quad \xi_4 = \frac{\partial}{\partial t}.$$

We easily get

$$[\xi_1, \xi_2] = \xi_3$$

and the other brackets are zero.

The dual frame of $\{\xi_1, \dots, \xi_4\}$ is given by

$$\alpha_1 = dx, \quad \alpha_2 = dy, \quad \alpha_3 = dz - xdy, \quad \alpha_4 = dt.$$

We have

$$d\alpha_1 = d\alpha_2 = d\alpha_4 = 0, \quad d\alpha_3 = -\alpha_1 \wedge \alpha_2.$$

The special symplectic structure on M is given by the forms

$$\begin{cases} \omega = \alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_4 \\ \psi = i(\alpha_1 + i\alpha_3) \wedge (\alpha_2 + i\alpha_4) \end{cases}$$

and by the almost complex structure

$$\begin{aligned} J(\xi_1) &= \xi_3, & J(\xi_2) &= \xi_4, \\ J(\xi_3) &= -\xi_1, & J(\xi_4) &= -\xi_2. \end{aligned}$$

We immediately get

$$\begin{aligned} \Im \psi &= \alpha_1 \wedge \alpha_2 - \alpha_3 \wedge \alpha_4, \\ \Re \psi &= \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_4. \end{aligned}$$

Hence

$$d\Re \psi = 0.$$

Let $X \subset G$ be the set

$$X = \{A \in G \mid x = t = 0\}$$

and

$$L = \pi(X),$$

where $\pi : G \rightarrow M$ is the natural projection. Hence L is a compact manifold embedded in M . Moreover the tangent bundle to L is generated by $\{\xi_2, \xi_3\}$; so we get

$$\begin{aligned} p^*(\omega) &= 0, \\ p^*(\Im \psi) &= 0. \end{aligned}$$

Hence L is a special Lagrangian torus.

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REFERENCES

- [1] Chiossi S., Fino A.: Conformally parallel G_2 structures on a class of solvmanifolds. *Math. Z.* **252** (2006), pp. 825–848.
- [2] Chiossi S., Salamon S.: The intrinsic torsion of $SU(3)$ and G_2 structures, *Differential geometry*, Valencia, 2001, World Sci. Publishing, River Edge, NJ (2002) pp. 115–133.
- [3] Conti D., Tomassini A.: Special Symplectic 6-manifolds, to appear in *Q. J. Math.*, **e-print math.DG/0601002** (2006).
- [4] Conti D., Salamon S.: Generalized Killing spinors in dimension 5, e-print math.DG/0508375, to appear in *Trans. Amer. Math. Soc.*
- [5] de Bartolomeis P.: Geometric Structures on Moduli Spaces of Special Lagrangian Submanifolds, *Ann. di Mat. Pura ed Applicata*, IV, Vol. CLXXIX, (2001), pp. 361–382.
- [6] de Bartolomeis P., Tomassini A.: On the Maslov Index of Lagrangian Submanifolds of Generalized Calabi-Yau Manifolds, *Int. J. of Math.* **17**, (2006) pp. 921–947.

- [7] de Bartolomeis P., Tomassini A.: On solvable Generalized Calabi-Yau Manifolds, *Ann. Inst. Fourier* **56**, (2006) pp. 1281–1296.
- [8] Fernández M., de León M., Saralegui M.: A six dimensional Compact Symplectic Solvmanifold without Kähler Structures, *Osaka J. Math* **33** (1996) pp. 19–34.
- [9] Fernández M., Gray A.: Compact symplectic sovmanifold not admitting complex structures, *Geometria Dedicata* **34** (1990), pp. 295–299.
- [10] Gauduchon P.: Hermitian connections and Dirac operators. *Boll. Un. Mat. Ital. B* (7) **11** (1997), no. 2, suppl., pp. 257–288.
- [11] Gualtieri M.: Generalized complex geometry, DPhil thesis, University of Oxford, 2003, e-print [math.DG/0401221](https://arxiv.org/abs/math.DG/0401221).
- [12] Harvey R., Lawson H. Blaine Jr.: Calibrated geometries, *Acta Math.* **148** (1982) pp. 47–157.
- [13] Hitchin N.J.: *Stable forms and special metrics*, Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), 70–89, Contemp. Math., 288, Amer. Math. Soc., Providence, RI, 2001.
- [14] Hitchin N.J.: Generalized Calabi-Yau Manifolds, *Quart. J. Math.* **54** (2003) pp. 281–308.

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